WEIGHTED ERGODIC THEOREMS FOR BANACH-KANTOROVICH LATTICE $L_p(\widehat{\nabla}, \widehat{\mu})$

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Abstract. In the present paper we prove weighted ergodic theorems and multi-parameter weighted ergodic theorems for positive contractions acting on $L_p(\hat{\nabla}, \hat{\mu})$. Our main tool is the use of methods of measurable bundles of Banach-Kantorovich lattices.

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1. Introduction

The present paper is devoted to the weighted ergodic theorems for positive contractions acting on Banach-Kantorovich lattice $L_p(\hat{\nabla}, \hat{\mu})$. Note that in [22, 1] weighted ergodic theorems for Danford-Schwraz operators acting on L_p-spaces were proved. Further, in [2, 3] such results were extended to Banach-valued functions. In [23] it has been considered weighted ergodic theorems and strong laws of large numbers. In [20] some properties of the convergence of Banach-valued martingales were described and their connections with the geometrical properties of Banach spaces were established too.

It is known that the theory of Banach bundles stemming from the paper [13], where it was proved such a theory has vast applications in analysis. In [12, 13, 16, 17]) the theory of Banach–Kantorovich spaces were developed. In [8] Banach-Kantorovich lattice $L_p(\widehat{\nabla}, \widehat{\mu})$ is represented as a measurable bundle of classical L_p –lattices. Hence, with the development of the theory Banach–Kantorovich spaces there naturally arises the necessity to study some ergodic type theorems for positive contractions and martingales defined on such spaces. In [5] an analog of the individual ergodic theorem for positive contractions of Banach-Kantorovich lattices $L_p(\widehat{\nabla}, \widehat{\mu})$, has bee established. In [21] such a result has been extended to Orlich-Kantorovich lattices. In [7] the convergence of martingales on such lattices is proved. Further, in [9] the "zero-two" law for positive contractions of Banach-Kantorovich lattice $L_p(\widehat{\nabla}, \widehat{\mu})$ has been proved.

In the present paper we are going to prove weighted ergodic theorems and multi-parameter weighted ergodic theorems for positive contractions acting on $L_p(\hat{\nabla}, \hat{\mu})$. We note that more effective methods to study of Banach-Kantorovich spaces are the methods of Boolean-valued analysis and measurable bundles (see [16],[17], [18]). In the present paper we shall use the methods of measurable bundles of Banach-Kantorovich lattices.

2. Preliminaries

In this section we recall and formulate necessary definitions and results about Banach-Kantorovich lattices.

Let $(\Omega, \Sigma, \lambda)$ be a measurable space with finite measure λ, and $L_0(\Omega)$ be the algebra of all measurable functions on Ω (here the functions equal a.e. are identified) and let $\nabla(\Omega)$ be the Boolean algebra of all idempotents in $L_0(\Omega)$. By ∇ we denote an arbitrary complete Boolean subalgebra of $\nabla(\Omega)$. By $\mathcal{L}^\infty(\Omega)$ we denote the set of all measurable essentially bounded functions on Ω, and $L^\infty(\Omega)$ denote an algebra of equivalence classes of essentially bounded measurable functions.

Let E be a linear space over the real field $\mathbb{R}$. By $\|\cdot\|$ we denote a $L_0(\Omega)$-valued norm on E. Then the pair $(E, \|\cdot\|)$ is called a *lattice-normed space (LNS) over* $L_0(\Omega)$. An LNS E is said to be *d-decomposable* if for every $x \in E$ and the decomposition $\|x\| = f + g$ with f and g disjoint positive elements in $L_0(\Omega)$ there exist $y, z \in E$ such that $x = y + z$ with $\|y\| = f$, $\|z\| = g$.

Suppose that $(E, \|\cdot\|)$ is an LNS over $L_0(\Omega)$. A net $\{x_\alpha\}$ of elements of E is said to be (bo)*-converging* to $x \in E$ (in this case we write $x = (bo)\text{-}\lim x_\alpha$), if the net $\{\|x_\alpha - x\|\}$ (o)-converges to zero in $L_0(\Omega)$ (written as $(o)\text{-}\lim \|x_\alpha - x\| = 0$). A net $\{x_\alpha\}_{\alpha \in A}$ is called (bo)*-fundamental* if $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero.

An LNS in which every (bo)-fundamental net (bo)-converges is called (bo)*-complete*. A *Banach-Kantorovich space (BKS) over* $L_0(\Omega)$ is a (bo)-complete d-decomposable LNS over $L_0(\Omega)$. It is well known [16],[17] that every BKS E over $L_0(\Omega)$ admits an $L_0(\Omega)$-module structure such that $\|fx\| = |f| \cdot \|x\|$ for every $x \in E$, $f \in L_0(\Omega)$, where $|f|$ is the modulus of a function $f \in L_0(\Omega)$. A BKS $(\mathcal{U}, \|\cdot\|)$ is called a *Banach-Kantorovich lattice* if $\mathcal{U}$ is a vector lattice and the norm $\|\cdot\|$ is monotone, i.e. $|u_1| \leq |u_2|$ implies $\|u_1\| \leq \|u_2\|$. It is known [16] that the cone $\mathcal{U}_+$ of positive elements is (bo)-closed.

Let $(\Omega, \Sigma, \lambda)$ be the same as above and X be an assisting real Banach space $(X(\omega), \|\cdot\|_{X(\omega)})$ to each point $\omega \in \Omega$, where $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A *section* of X is a function u defined λ-almost everywhere in Ω that takes values $u(\omega) \in X(\omega)$ for all ω in the domain $dom(u)$ of u. Let L be a set of sections. The pair (X, L) is called a *measurable Banach bundle over* Ω if

(1) $\alpha_1 u_1 + \alpha_2 u_2 \in L$ for every $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u_1, u_2 \in L$, where $\alpha_1 u_1 + \alpha_2 u_2 : \omega \in dom(u_1) \cap dom(u_2) \to \alpha_1 u_1(\omega) + \alpha_2 u_2(\omega)$;
(2) the function $\|u\| : \omega \in dom(u) \to \|u(\omega)\|_{X(\omega)}$ is measurable for every $u \in L$;
(3) the set $\{u(\omega) : u \in L, \omega \in dom(u)\}$ is dense in $X(\omega)$ for every $\omega \in \Omega$.

A measurable Banach bundle (X, L) is called *measurable bundle of Banach lattices (MBBL)* if $(X(\omega), \|\cdot\|_{X(\omega)})$ is a Banach lattice for all $\omega \in \Omega$ and for every $u_1, u_2 \in L$ one has $u_1 \vee u_2 \in L$, where $u_1 \vee u_2$: $\omega \in \text{dom }(u_1) \cap \text{dom }(u_2) \to u_1(\omega) \vee u_2(\omega)$.

A section s is called *step-section* if it has a form

$$s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega) u_i(\omega),$$

for some $u_i \in L$, $A_i \in \Sigma$, $A_i \cap A_j = \emptyset$, $i \neq j$, $i, j = 1, \cdots, n$, $n \in \mathbb{N}$, where χ_A is the indicator of a set A. A section u is called *measurable* there exists a sequence of step-functions $\{s_n\}$ such that $s_n(\omega) \to u(\omega)$ λ-a.e.

By $M(\Omega, X)$ we denote the set all measurable sections, and by $L_0(\Omega, X)$ the factor space of $M(\Omega, X)$ with respect to the equivalence relation of the equality a.e. Clearly, $L_0(\Omega, X)$ is an $L_0(\Omega)$-module. The equivalence class of an element $u \in M(\Omega, X)$ is denoted by $\hat{u}$. The norm of $\hat{u} L_0(\Omega, X)$ is defined as a class of equivalence in $L_0(\Omega)$ containing the function $\|u(\omega)\|_{X(\omega)}$, namely $\|\hat{u}\| = (\widehat{\|u(\omega)\|_{X(\omega)}})$. In [12] it was proved that $L_0(\Omega, X)$ is a BKS over $L_0(\Omega)$. Furthermore, for every BKS E

over $L_0(\Omega)$ there exists a measurable Banach bundle (X, L) over Ω such that E is isomorphic to $L_0(\Omega)$.

Let X be a MBBL. We put $\hat{u} \leq \hat{v}$ if $u(\omega) \leq v(\omega)$ a.e. One can see that the relation $\hat{u} \leq \hat{v}$ is a partial order in $L_0(\Omega, X)$. If X is a MBBL, then $L_0(\Omega, X)$ is a Banach-Kantarovich lattice [6, 8].

A mapping $\mu : \nabla \to L_0(\Omega)$ is called a $L_0(\Omega)$-*valued measure* if the following conditions are satisfied:

1) $\mu(e) \geq 0$ for all $e \in \nabla$;
2) if $e \wedge g = 0, e, g \in \nabla$, then $\mu(e \vee g) = \mu(e) + \mu(g)$;
3) if $e_n \downarrow 0,\ e_n \in \nabla,\ n \in \mathbb{N}$, then $\mu(e_n) \downarrow 0$.

A $L_0(\Omega)$-valued measure μ is called *strictly positive* if $\mu(e) = 0,\ e \in \nabla$ implies $e = 0$.

Let a Boolean algebra $\nabla(\Omega)$ of all idempotents of $L_0(\Omega)$ is a regular subalgebra of ∇.

In the sequel we will consider a strictly positive $L_0(\Omega)$-valued measure μ with the following property $\mu(ge) = g\mu(e)$ for all $e \in \nabla$ and $g \in \nabla(\Omega)$.

Let $\nabla_\omega,\ \omega \in \Omega$ be complete Boolean algebras with strictly positive real-valued measures μ_ω. Put $\rho_\omega(e, g) = \mu_\omega(e \bigtriangleup g),\ e, g \in \nabla_\omega$. Then $(\nabla_\omega, \mu_\omega)$ is a complete metric space. Let us consider a mapping ∇, which assigns to each $\omega \in \Omega$ a Boolean algebra ∇_ω. Such a mapping is called a section.

Assume that L is a nonempty set of sections ∇. A pair (∇, L) is called *a measurable bundle of Boolean algebras over* Ω if one has

1) (∇, L) is a measurable bundle of metric spaces (see [8]);
2) if $e \in L$, then $e^\perp \in L$, where $e^\perp : \omega \in \text{dom }(e) \to e^\perp(\omega)$;
3) if $e_1, e_2 \in L$, then $e_1 \vee e_2 \in L$, where $e_1 \vee e_2 : \omega \in \text{dom }(e_1) \cap \text{dom }(e_2) \to e_1(\omega) \vee e_2(\omega)$.

Let $M(\Omega, \nabla)$ be the set of all measurable sections, and $\hat{\nabla}$ be a factorization of $M(\Omega, \nabla)$ with respect to equivalence relation the equality a.e. Let us define a mapping $\hat{\mu} : \hat{\nabla} \to L_0(\Omega)$ by $\hat{\mu}(\hat{e}) = \hat{f}$, where $\hat{f}$ is a class containing the function $f(\omega) = \mu_\omega(e(\omega))$. It is clear that the mapping $\hat{\mu}$ is well-defined. It is known that $(\hat{\nabla}, \hat{\mu})$ is a complete Boolean algebra with a strictly positive $L_0(\Omega)$-valued measure $\hat{\mu}$. Note that a Boolean algebra $\nabla(\Omega)$ of all idempotents of $L_0(\Omega)$ is identified with a regular subalgebra of $\hat{\nabla}$, and one has $\widehat{\mu}(g\widehat{e}) = g\widehat{\mu}(\widehat{e})$ for all $g \in \nabla(\Omega)$ and $\widehat{e} \in \hat{\nabla}$.

The reverse is also tree, namely one has the following

Theorem 2.1. [8] *Let $\tilde{\nabla}$ be a complete Boolean algebra, $\tilde{\mu}$ be a strictly positive $L_0(\Omega)$-valued measure on $\tilde{\nabla}$, and $\nabla(\Omega)$ is a regular subalgebra of $\tilde{\nabla}$ and $\tilde{\mu}(g\tilde{e}) = g\tilde{\mu}(\tilde{e})$ for all $g \in \nabla(\Omega)$, $\tilde{e} \in \tilde{\nabla}$. Then there exists a measurable bundle of Boolean algebras (∇, L) such that $\hat{\nabla}$ is isometrically isomorphic to $\tilde{\nabla}$.*

By the equality $\mu(e) = \frac{\tilde{\mu}(e)}{1+\tilde{\mu}(\mathbb{1})}$ we define $L^\infty(\Omega)$-valued strictly positive measure on $\tilde{\nabla}$, where $\mathbb{1}$ is an identity in $\tilde{\nabla}$.

Assume that p is a lifting on $L^\infty(\Omega)$ [13]. Define a real-valued quasi-measure on $\tilde{\nabla}$ by $\mu^0_\omega(e) = p(\mu(e))(\omega)$ for all $\omega \in \Omega$.

Let $I^0_\omega = \{e \in \tilde{\nabla} : \mu^0_\omega(e) = 0\}$ for all $\omega \in \Omega$. It is clear that I^0_ω is an ideal of $\tilde{\nabla}$. Put $\nabla^0_\omega = \tilde{\nabla}/I^0_\omega$. Then ∇^0_ω is a Boolean algebra with strictly positive quasi-measure μ^0_ω. Let us complete the metric space $(\nabla^0_\omega, \rho_\omega)$, where $\rho_\omega(e, g) = \mu^0_\omega(e \bigtriangleup g)$, and completion we denote by ∇_ω. Then ∇_ω is a complete Boolean algebra with strictly positive real-valued measure μ_ω, which is an extension of μ^0_ω.

Assume that $\pi_\omega : \tilde{\nabla} \to \nabla^0_\omega$ is a factor-homomorphism, $i_\omega : \nabla^0_\omega \to \nabla_\omega$ is the inclusion, then $\gamma_\omega = i_\omega \circ \pi_\omega$ is a homomorphism from the Boolean algebra $\tilde{\nabla}$ into Boolean algebra ∇_ω.

Let ∇ be a mapping, which assigns to each $\omega \in \Omega$ a Boolean algebra ∇_ω with strictly positive real-valued measure μ_ω, and $L = \{\tilde{e} : \tilde{e}(\omega) = \gamma_\omega(e), e \in \tilde{\nabla}\}$. It is known [8] that the pair (∇, L) is a measurable bundle of Boolean algebras and $\hat{\nabla}$ is isometrically isomorphic to $\tilde{\nabla}$. Moreover, one has $\widehat{\mu} = \tilde{\mu}$ (see [8] for more details).

By $L_0(\hat{\nabla}, \hat{\mu})$ we denote an order complete vector lattice $C_\infty(Q(\hat{\nabla}))$, where $Q(\hat{\nabla})$ is the Stonian compact associated with complete Boolean algebra $\hat{\nabla}$. For $\widehat{f}, \widehat{g} \in L_0(\hat{\nabla}, \hat{\mu})$ we let $\widehat{\rho}(\widehat{f}, \widehat{g}) = \int \frac{|\hat{f}-\hat{g}|}{1+|\hat{f}-\hat{g}|} d\hat{\mu}$. Then it is know [8] that $\widehat{\rho}$ is an $L_0(\Omega)$-valued metric on $L_0(\hat{\nabla}, \hat{\mu})$ and $(L_0(\hat{\nabla}, \hat{\mu}), \widehat{\rho})$ is isometrically isomorphic to the measurable bundle of metric spaces $L_0(\nabla_\omega, \mu_\omega)$, where $\rho_\omega(a, b) = \int \frac{|a-b|}{1_\omega+|a-b|} d\mu_\omega$. In particularly, each element $\widehat{f} \in L_0(\hat{\nabla}, \hat{\mu})$ can be identified with the measurable section $\{f(\omega)\}_{\omega\in\Omega}$, here $f(\omega) \in L_0(\nabla_\omega, \mu_\omega)$.

Following the well known scheme of the construction of L_p-spaces, a space $L_p(\nabla, \mu)$ can be defined by

$$L_p(\hat{\nabla}, \hat{\mu}) = \left\{ \hat{f} \in L_0(\hat{\nabla}) : \int |\hat{f}|^p d\hat{\mu} - \text{exist} \right\}, \quad p \geq 1$$

where $\hat{\mu}$ is a $L_0(\Omega)$-valued measure on $\hat{\nabla}$.

It is known [16] that $L_p(\hat{\nabla}, \hat{\mu})$ is a BKS over $L_0(\Omega)$ with respect to the $L_0(\Omega)$-valued norm $\|\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})} = \left(\int |\hat{f}|^p d\hat{\mu} \right)^{1/p}$. Moreover, $L_p(\hat{\nabla}, \hat{\mu})$ is a Banach-Kantorovich lattice (see [17],[8]).

Let X be a mapping assisting an L_p-space constructed by a real-valued measure μ_ω, i.e. $L_p(\nabla_\omega, \mu_\omega)$ to each point $\omega \in \Omega$ and let

$$L = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{R}, \quad e_i \in M(\Omega, \nabla), \; i = \overline{1, n}, \; n \in \mathbb{N} \right\}$$

be a set of sections. In [8, 10] it has been established that the pair (X, L) is a measurable bundle of Banach lattices and $L_0(\Omega, X)$ is modulo ordered isomorphic to $L_p(\hat{\nabla}, \hat{\mu})$.

Let as before $p \geq 1$ and $L_p(\hat{\nabla}, \hat{\mu})$ be a Banach-Kantorovich lattice, and $L_p(\nabla_\omega, \mu_\omega)$ be the corresponding L_p-spaces constructed by a real valued measures. Let $T : L_p(\hat{\nabla}, \hat{\mu}) \to L_p(\hat{\nabla}, \hat{\mu})$ be a linear mapping. As usual we will say that T is *positive* if $T\hat{f} \geq 0$ whenever $\hat{f} \geq 0$. We say that T is a $L_0(\Omega)$*-bounded mapping* if there exists a function $k \in L_0(\Omega)$ such that $\|T\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})} \leq k\|\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})}$ for all $\hat{f} \in L_p(\hat{\nabla}, \hat{\mu})$. For a such mapping we can define an element of $L_0(\Omega)$ as follows

$$\|T\| = \sup_{\|\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})} \leq \mathbb{1}} \|T\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})},$$

which is called an $L_0(\Omega)$*-valued norm* of T. If $\|T\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})} \leq \|\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})}$ then a mapping T is said to be a $L_p(\hat{\nabla}, \hat{\mu})$ *contraction*.

The set of all essentially bounded functions w.r.t. $\hat{f}$ taken from $L_0(\hat{\nabla}, \hat{\mu})$ is denoted by $L^\infty(\hat{\nabla}, \hat{\mu})$.

Let $\hat{\nabla}^{(1)}$ be a regular Boolean subalgebra of $\hat{\nabla}$, and $\hat{\mu}^1$ is the restriction of $\hat{\mu}$ onto $\hat{\nabla}^1$. Then according to Theorem 4.2.9 [16] there exists the *conditional expectation* operator $E(\cdot|\hat{\nabla}^1) : L_1(\hat{\nabla}, \hat{\mu}) \to L_1(\hat{\nabla}^{(1)}, \hat{\mu}^1)$ which satisfies the following conditions:

(a) $E(\cdot|\hat{\nabla}^{(1)})$ is linear, positive and idempotent;
(b) for every $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$ one has $\int E(\hat{f}|\hat{\nabla}^{(1)})d\hat{\mu} = \int \hat{f} d\hat{\mu}$;
(c) $E(\hat{g}\hat{f}|\hat{\nabla}^{(1)}) = \hat{g}E(\hat{f}|\hat{\nabla}^{(1)})$ for every $\hat{g} \in L^{\infty}(\hat{\nabla}^{(1)}, \hat{\mu}^{(1)})$ and $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$.
(d) Moreover, one has $\|E(\hat{f}|\hat{\nabla}^{(1)})\|_{L_1(\hat{\nabla},\hat{\mu})} \leq \|\hat{f}\|_{L_1(\hat{\nabla},\hat{\mu})}$ for every $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$ and $E(\mathbb{1}|\hat{\nabla}^{(1)}) = \mathbb{1}$.

In the sequel we will need the following

Theorem 2.2. [9, 8] *Let $T : L_p(\hat{\nabla}, \hat{\mu}) \to L_p(\hat{\nabla}, \hat{\mu})$ be a positive linear $L_p(\hat{\nabla}, \hat{\mu})$ contraction such that $T\mathbb{1} \leq \mathbb{1}$. Then for every $\omega \in \Omega$ there exists a positive $L_p(\nabla_\omega, \mu_\omega)$ contraction $T(\omega) : L_p(\nabla_\omega, \mu_\omega) \to L_p(\nabla_\omega, \mu_\omega)$ such that $T(\omega)f(\omega) = (T\hat{f})(\omega)$ λ-a.e. for every $\hat{f} \in L_p(\hat{\nabla}, \hat{\mu})$.*

By means of measurable bundle of L_p-spaces and at each bundle applying classical ergodic theorem, it has been proved the following

Theorem 2.3. [5] *Let $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $T : L_p(\hat{\nabla}, \hat{\mu}) \to L_p(\hat{\nabla}, \hat{\mu})$ be a linear positive $L_p(\hat{\nabla}, \hat{\mu})$ contraction with $T\mathbb{1} \leq \mathbb{1}$. Then for every $\hat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ one has*

(i) *the sequence*
$$s_n(|\hat{f}|) = \frac{1}{n}\sum_{i=0}^{n-1} T^i(|\hat{f}|)$$
is bounded in $L_p(\hat{\nabla}, \hat{\mu})$, and
$$\|\sup_{n\geq 1} s_n(|\hat{f}|)\|_{L_p(\hat{\nabla},\hat{\mu})} \leq q\|\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})};$$
(ii) *there exists an element $\tilde{f} \in L_p(\hat{\nabla}, \hat{\mu})$ such that the sequence $s_n(\hat{f})$ (o)-converges to $\tilde{f}$ in $L_p(\hat{\nabla}, \hat{\mu})$.*

It is worth to mention that in [21] it has been proved that for every $\hat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ at $p \geq 1$ the averages $s_n(\hat{f})$ (bo)-converge in $L_p(\hat{\nabla}, \hat{\mu})$.

3. (o)-convergence

In this section we provide some auxiliary facts related to (o)-convergence of sequence $\hat{f}_{\mathbf{n}}$ from $L_0(\hat{\nabla}, \hat{\mu})$ and (o)-convergence of the sequence $\{f_{\mathbf{n}}(\omega)\}$ from $L_0(\nabla_\omega, \mu_\omega)$.

For $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ denote $m(\mathbf{n}) = \min\{n_1, \dots, n_d\}$. In the sequel by $\mathbf{n} \to \infty$ we mean $m(\mathbf{n}) \to \infty$. For $\mathbf{n} = (n_1, n_2, \dots, n_d), \mathbf{k} = (k_1, k_2, \dots, k_d)$ we write $\mathbf{n} \leq \mathbf{k}$ if and only if $n_i \leq k_i$ (i=1,2, … ,d), $\mathbf{n} < \mathbf{k}$ if $n_i < k_i$ (i=1,2, … ,d).

Theorem 3.1. *Let $\widehat{f_{\mathbf{n}}} \in L_0(\hat{\nabla}, \hat{\mu})$. Then $\sup\limits_{\mathbf{n}} \hat{f}_{\mathbf{n}}$ exists in $L_0(\hat{\nabla}, \hat{\mu})$ if and only if $\sup\limits_{\mathbf{n}} f_{\mathbf{n}}(\omega)$ exists in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. In the later case, one has $(\sup\limits_{\mathbf{n}} \hat{f}_{\mathbf{n}})(\omega) = \sup\limits_{\mathbf{n}} f_{\mathbf{n}}(\omega)$ for a.e. $\omega \in \Omega$.*

Proof. Assume that $g(\omega) = \sup\limits_{\mathbf{n}} f_{\mathbf{n}}(\omega)$ exists in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. Denote $\hat{g}_{\mathbf{n}} = \sup\limits_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}} \hat{f}_{\mathbf{k}}$ in $L_0(\hat{\nabla}, \hat{\mu})$. Then $g_{\mathbf{n}}(\omega) = \sup\limits_{\mathbf{1}\leq\mathbf{k}\leq\mathbf{n}} f_{\mathbf{k}}(\omega)$ for a.e. $\omega \in \Omega$.

Obviously, that $g_{\mathbf{n}}(\omega) \uparrow g(\omega)$ as $\mathbf{n} \to \infty$ for a.e. $\omega \in \Omega$. The relation $g_{\mathbf{n}}(\omega) \uparrow g(\omega)$ implies that $g_{\mathbf{n}}(\omega) \stackrel{\rho_\omega}{\to} g(\omega)$ for a.e. $\omega \in \Omega$, this means $g \in M(\Omega, X)$ and $\hat{g} \in L_0(\hat{\nabla}, \hat{\mu})$.

Let us prove that $\hat{g} = \sup_{\mathbf{n}} \hat{f}_{\mathbf{n}}$ in $L_0(\hat{\nabla}, \hat{\mu})$. It is clear that $g(\omega) \geq f_{\mathbf{n}}(\omega)$ for a.e. $\omega \in \Omega$. Therefore, $\widehat{g} \geq \widehat{f_{\mathbf{n}}}$ for all $\mathbf{n} \in \mathbb{N}^d$.

Let $\widehat{\varphi} \in L^0(\hat{\nabla}, \hat{\mu})$ and $\widehat{\varphi} \geq \widehat{f_{\mathbf{n}}}$ for all $\mathbf{n} \in \mathbb{N}^d$. Then $\varphi(\omega) \geq f_{\mathbf{n}}(\omega)$ for any $\mathbf{n} \in \mathbb{N}^d$. Hence, $\varphi(\omega) \geq g(\omega)$, for a.e. $\omega \in \Omega$, i.e. $\hat{\varphi} \geq \hat{g}$. This yields that $\hat{g} = \sup_{\mathbf{n} \in \mathbb{N}^d} \hat{f}_{\mathbf{n}}$.

Conversely, let us assume that there exists such $\hat{\psi} \in L_0(\hat{\nabla}, \hat{\mu})$ that $\hat{\psi} = \sup_{\mathbf{n} \geq \mathbf{1}} \hat{f}_{\mathbf{n}} = \sup_{\mathbf{n} \in \mathbb{N}^d} \hat{g}_{\mathbf{n}}$.

From $g_{\mathbf{n}}(\omega) = \sup_{1 \leq \mathbf{k} \leq \mathbf{n}} f_{\mathbf{k}}(\omega)$ for a.e. $\omega \in \Omega$, we find $\psi(\omega) \geq f_{\mathbf{n}}(\omega)$ for all $\mathbf{n} \in \mathbb{N}^d$ for a.e. $\omega \in \Omega$. Hence, one gets $\psi(\omega) \geq \sup_{\mathbf{n} \in \mathbb{N}^d} f_{\mathbf{n}}(\omega) = \sup_{\mathbf{n} \in \mathbb{N}^d} g_{\mathbf{n}}(\omega)$ for a.e. $\omega \in \Omega$. As $\hat{g}_{\mathbf{n}} \to \hat{\psi}$ in metric $\hat{\rho}$, then $g_{\mathbf{n}}(\omega) \to \psi(\omega)$ in metric ρ_ω for a.e. $\omega \in \Omega$. Since $\{g_{\mathbf{n}}(\omega)\}$ is increasing then $\psi(\omega) = \sup_{\mathbf{n} \in \mathbb{N}^d} g_{\mathbf{n}}(\omega)$ for a.e. $\omega \in \Omega$. $\square$

From this theorem immediately follows two corollaries.

Corollary 3.2. *Let $\{\hat{f}_{\mathbf{n}}\} \subset L_0(\hat{\nabla}, \hat{\mu})$. Then $\inf_{\mathbf{n} \in \mathbb{N}^d} \hat{f}_{\mathbf{n}}$ exists in $L_0(\hat{\nabla}, \hat{\mu})$ if and only if $\inf_{\mathbf{n} \in \mathbb{N}^d} f(\omega)$ exists in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. In later case, one has $(\inf_{\mathbf{n} \in \mathbb{N}^d} \hat{f}_{\mathbf{n}})(\omega) = \inf_{\mathbf{n} \in \mathbb{N}^d} f_{\mathbf{n}}(\omega)$ for a.e. $\omega \in \Omega$.*

Corollary 3.3. *Let $\hat{f}_{\mathbf{n}} \in L_0(\hat{\nabla}, \hat{\mu})$. If $\hat{f}_{\mathbf{n}} \stackrel{(o)}{\to} \hat{f}$ for some $\hat{f} \in L_0(\hat{\nabla}, \hat{\mu})$, then $f_{\mathbf{n}}(\omega) \stackrel{(o)}{\to} f(\omega)$ in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. Conversely, if $f_{\mathbf{n}}(\omega) \stackrel{(o)}{\to} g(\omega)$ for some $g(\omega) \in L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$, then $\hat{g} \in L_0(\hat{\nabla}, \hat{\mu})$ and $\hat{f}_{\mathbf{n}} \stackrel{(o)}{\to} \hat{g}$ in $L_0(\hat{\nabla}, \hat{\mu})$.*

4. Weighted ergodic theorems

In this section we shall prove some weighted ergodic theorems in $L_p(\hat{\nabla}, \hat{\mu})$.

First we recall that a sequence $\{\alpha(k)\}$ is called *Besicovich* if for every $\varepsilon > 0$ there is a sequence of trigonometric polynomials ψ_ε, such that

$$\lim_{N \to \infty} \sup \frac{1}{N} \sum_{k=1}^{N-1} |\alpha(k) - \psi_\varepsilon(k)| < \varepsilon$$

We say that $\{\alpha(k)\}$ is *bounded Besicovich* if $\alpha(k) \in \ell^\infty$. In what follows, we consider only bounded, real Besicovich sequences.

Theorem 4.1. *Let $T : L_1(\hat{\nabla}, \hat{\mu}) \to L_1(\hat{\nabla}, \hat{\mu})$ be a positive linear $L_1(\hat{\nabla}, \hat{\mu})$ contraction with $T\mathbb{1} \leq \mathbb{1}$, and $\{\alpha(k)\}$ be a bounded Besicovich sequence. Then for every $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$ the averages*

$$\widetilde{A_N}(\widehat{f}) = \frac{1}{N} \sum_{k=1}^{N-1} \alpha(k) T^k \widehat{f}$$

(o)-converge in $L_0(\widehat{\nabla}, \widehat{\mu})$.

Proof. According to Theorem 2.2 for each $\omega \in \Omega$ there exists a positive contraction $T_\omega : L_1(\nabla_\omega, \mu_\omega) \to L_1(\nabla_\omega, \mu_\omega)$, such that $T_\omega f(\omega) = (T\hat{f})(\omega)$ for every $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$ and a.e. $\omega \in \Omega$. Then for every $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$ we have

$$\begin{aligned} \widetilde{A_N}(\widehat{f})(\omega) &= (\frac{1}{N}\sum_{k=1}^{N-1} \alpha(k) T^k \widehat{f})(\omega) \\ &= \frac{1}{N}\sum_{k=1}^{N-1} \alpha(k) T^k(\omega) f(\omega) = A_N(\widehat{f}(\omega)) \end{aligned} \tag{1}$$

for a.e. $\omega \in \Omega$.

By means of Theorem 1.4 [1] one gets the existence of the limit

$$(o) - \lim \frac{1}{N}\sum_{k=1}^{N-1} \alpha(k) T^k(\omega) f(\omega)$$

for every $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$, and the limit belongs to $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. This means

$$\widetilde{A_N}(\widehat{f})(\omega) = A_N(\widehat{f}(\omega)) \xrightarrow{(o)} \widetilde{f}(\omega)$$

in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$ and for some $\widetilde{f}(\omega) \in L_0(\nabla_\omega, \mu_\omega)$. Due to Corollary 3.3 we obtain

$$\widetilde{A_N}(\widehat{f}) = \widehat{A_N(\widehat{f}(\omega))} \xrightarrow{(o)} \widehat{f} = \widehat{\widetilde{f}(\omega)}$$

in $L_0(\widehat{\nabla}, \widehat{\mu})$. □

Remark 4.2. *In case $\alpha(k) = 1$ Theorem 4.1 implies Theorem 3.2 (i)* [21]

Corollary 4.3. *Let $T : L_1(\hat{\nabla}, \hat{\mu}) \to L_1(\hat{\nabla}, \hat{\mu})$ be a positive linear $L_1(\hat{\nabla}, \hat{\mu})$ contraction with $T\mathbb{1} \leq \mathbb{1}$, and j_k be an increasing sequence of positive integers such that $\sup\limits_k \frac{j_k}{k} < \infty$. Then for every $\hat{f} \in L_1(\hat{\nabla}, \hat{\mu})$ the limit*

$$(o) - \lim \frac{1}{N}\sum_{k=1}^{N-1} T^{j_k} \widehat{f}$$

exists in $L_0(\widehat{\nabla}, \widehat{\mu})$.

Proof. Let us define

$$\alpha(k) = \begin{cases} 0, & \text{if } k \neq j_k; \\ 1, & \text{if } k = j_k. \end{cases}$$

Then is known [19] that $\alpha(k)$ is a Besicovich sequence. Hence, Theorem 4.1 implies the assertion. □

Theorem 4.4. *Let $p > 1,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $T : L_p(\hat{\nabla}, \hat{\mu}) \to L_p(\hat{\nabla}, \hat{\mu})$ be a positive linear $L_1(\hat{\nabla}, \hat{\mu})$ contraction with $T\mathbb{1} \leq \mathbb{1}$, and $\{\alpha(k)\}$ be a bounded Besicovich sequence. Then for every $\hat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ one has*

(i) *the sequence $A_N(|\hat{f}|)$ is bounded in $L_p(\hat{\nabla}, \hat{\mu})$ and*

$$\| \sup_{N \geq 1} A_N(|\hat{f}|) \|_{L_p(\widehat{\nabla}, \widehat{\mu})} \leq q \sup_k |\alpha(k)| \|\hat{f}\|_{L_p(\widehat{\nabla}, \widehat{\mu})};$$

(ii) *there exists an element $\tilde{f} \in L_p(\hat{\nabla}, \hat{\mu})$, such that the sequence $A_N(\hat{f})$ (o)-converges to $\tilde{f}$ in $L_p(\hat{\nabla}, \hat{\mu})$.*

Proof. (i) Due to Theorem 2.2 for each $\hat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ and for a.e. $\omega \in \Omega$ one has $\widetilde{A_N}(\widehat{f})(\omega) = A_N(f(\omega))$ (see (1)). Applying Akcoglu's Theorem [15] we have, that $\sup\limits_{N\geq 1} A_N(|f(\omega)|) \in L_p(\nabla_\omega, \mu_\omega)$ and

$$\|\sup_{N\geq 1} A_N(|f(\omega)|)\|_{L_p(\nabla_\omega,\mu_\omega))} \leq q \sup_k |\alpha(k)| \|f(\omega)\|_{L_p(\nabla_\omega,\mu_\omega)}$$

for a.e. $\omega \in \Omega$. Then we get

$$\begin{aligned} \|\sup_{N\geq 1} A_N(|\hat{f}|)\|_{L_p(\widehat{\nabla},\widehat{\mu})} &= \|\sup_{N\geq 1} A_N(|\widehat{f(\omega)}|)\|_{L_p(\nabla_\omega,\mu_\omega))} \\ &\leq q \sup_k |\alpha(k)| \|\widehat{f(\omega)}\|_{L_p(\nabla_\omega,\mu_\omega)} \\ &= q \sup_k |\alpha(k)| \|\widehat{f}\|_{L_p(\widehat{\nabla},\widehat{\mu})}. \end{aligned}$$

(ii) Since, T is positive linear $L_1(\hat{\nabla}, \hat{\mu})$ contraction in $L_p(\nabla_\omega, \mu_\omega)$ and $T\mathbb{1} \leq \mathbb{1}$, then by the Theorem 2.2 $T(\omega)$ is $L_1(\nabla_\omega, \mu_\omega))$ contraction in $L_p(\nabla_\omega, \mu_\omega))$ and $T(\omega)\mathbb{1}(\omega) \leq \mathbb{1}(\omega)$. This means that $T(\omega)$ is $L_1(\nabla_\omega, \mu_\omega)) - L_\infty(\nabla_\omega, \mu_\omega))$ contraction $L_p(\nabla_\omega, \mu_\omega))$ for a.e. $\omega \in \Omega$. From Theorem 1.2 [2] we find that the sequence $A_N(f(\omega))$ (o)- converges to some limit $\widetilde{f}(\omega)$ a.e. $\omega \in \Omega$, for every $f(\omega) \in L_p(\nabla_\omega, \mu_\omega)$. Then Corollary 3.3 implies that

$$\widetilde{A_N}(\widehat{f}) = \widehat{A_N(f(\omega))} \xrightarrow{(o)} \widehat{\widetilde{f}(\omega)} = \widetilde{f}$$

in $L_0(\widehat{\nabla}, \widehat{\mu})$. Due to

$$\|\sup_{N\geq 1} A_N(|\hat{f}|)\|_{L_p(\widehat{\nabla},\widehat{\mu})} \leq q \sup_k |\alpha(k)| \|\widehat{f}\|_{L_p(\widehat{\nabla},\widehat{\mu})}$$

one finds

$$\sup_{N\geq 1} A_N(|\hat{f}|) \in L_p(\widehat{\nabla}, \widehat{\mu}).$$

Therefore,

$$A_N(\hat{f}) \xrightarrow{(o)} \widetilde{f}$$

in $L_p(\widehat{\nabla}, \widehat{\mu})$. □

5. Multiparameter weighted ergodic theorems

In what follows, given $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ we denote $|\mathbf{n}| = n_1 \cdot n_2 \cdots n_d$, and $\mathbf{1} = (1, 1, \ldots, 1)$.

Let $T_1, T_2, \ldots, T_d$ be d linear positive $L_p(\hat{\nabla}, \hat{\mu})$ contractions in $L_p(\hat{\nabla}, \hat{\mu})$, $1 < p < \infty$. Then we denote $\mathbf{T}^{\mathbf{n}} = T_1^{n_1} \cdots T_d^{n_d}$, where $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$.

Theorem 5.1. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_d)$ denoted d linear positive $L_p(\hat{\nabla}, \hat{\mu})$ contractions in $L_p(\hat{\nabla}, \hat{\mu})$, $1 < p < \infty$ such that $T_i\mathbb{1} \leq \mathbb{1}$ for all $i : 1 \leq i \leq d$. Then for every $\widehat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ one has*

(i) *The averages*

$$S_{\mathbf{n}}(|\widehat{f}|) = \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} \mathbf{T}^{\mathbf{k}}(|\widehat{f}|)$$

is bounded in $L_p(\hat{\nabla}, \hat{\mu})$, and one has

$$\|\sup_{\mathbf{n}} S_{\mathbf{n}}(|\widehat{f}|)\|_{L_p(\widehat{\nabla},\widehat{\mu})} \leq q^d \|\widehat{f}\|_{L_p(\widehat{\nabla},\widehat{\mu})};$$

(ii) *There exists an element* $\widetilde{f} \in L_p(\hat{\nabla}, \hat{\mu})$, *such that* $S_{\mathbf{n}}(\widehat{f})$ (o)*- convergence to* $\widetilde{f}$ *in* $L_p(\hat{\nabla}, \hat{\mu})$.

Proof. Due to Theorem 2.2 one has

$$\begin{aligned} S_{\mathbf{n}}(\widehat{f})(\omega) &= \frac{1}{|\mathbf{n}|}\sum_{k_1=1}^{n_1}\cdots\sum_{k_d=1}^{n_d} T_1^{k_1}\cdots T_d^{k_d}(\widehat{f}))(\omega) \\ &= \frac{1}{|\mathbf{n}|}\sum_{k_1=1}^{n_1}\cdots\sum_{k_d=1}^{n_d} T_1^{k_1}(\omega)\cdots T_d^{k_d}(\omega)(\widehat{f}(\omega)) \\ &= S_{\mathbf{n}}(\omega)(f(\omega)) \end{aligned}$$

for a.e. $\omega \in \Omega$.

By Theorem 1.2 [15] (p.196) we have $\|\sup_{\mathbf{n}} S_{\mathbf{n}}(\omega)(|f(\omega)|)\|_{L_p(\widehat{\nabla},\widehat{\mu})} \leq q^d \|f(\omega)\|_{L_p(\widehat{\nabla},\widehat{\mu})}$ and there is $g(\omega) \in L_p(\nabla_\omega, \mu_\omega)$ such that $S_{\mathbf{n}}(\omega)(f(\omega)) \stackrel{(o)}{\to} g(\omega)$ in $L_p(\nabla_\omega, \mu_\omega)$. According Theorem 3.1 $\sup_{\mathbf{n}} S_{\mathbf{n}}(\widehat{f})$ exists in $L_p(\hat{\nabla}, \hat{\mu})$, and one has

$$\|\sup_{\mathbf{n}} S_{\mathbf{n}}(|\widehat{f}|)\|_{L_p(\widehat{\nabla},\widehat{\mu})} \leq q^d \|\widehat{f}\|_{L_p(\widehat{\nabla},\widehat{\mu})}.$$

By Corollary 3.3 we obtain that

$$S_{\mathbf{n}}(\widehat{f}) = S_{\mathbf{n}}\widehat{(\omega)(f(\omega))} \stackrel{(o)}{\to} \widehat{g(\omega)} = \widetilde{f}$$

in $L_0(\hat{\nabla}, \hat{\mu})$. Since $S_{\mathbf{n}}(\widehat{f})$ is bounded in $L^p(\hat{\nabla}, \hat{\mu})$, then $S_{\mathbf{n}}(\widehat{f}) \stackrel{(o)}{\to} \widetilde{f}$ in $L_p(\hat{\nabla}, \hat{\mu})$. $\square$

The next theorem is an analog of multiparameter weighted individual ergodic theorem in Banach–Kantorovich lattice $L_p(\hat{\nabla}, \hat{\mu})$.

Recall that (see [1]) a class of weights $\{\alpha(\mathbf{k}) : \mathbf{k} \in \mathbb{N}^d\}$ is called *Besicovich*, if for any $\varepsilon > 0$ there is sequence of trigonometric polynomials in d variables , ψ_ε such that

$$\limsup_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} |\alpha(\mathbf{k}) - \psi_\varepsilon(\mathbf{k})| < \varepsilon.$$

We say that $\{\alpha(\mathbf{k})\}$ is *bounded Besicovich* if $\alpha(\mathbf{k}) \in \ell^\infty$.

Theorem 5.2. *Let* $\mathbf{T} = (T_1, T_2, \dots, T_d)$ *denoted* d *linear positive* $L_1(\hat{\nabla}, \hat{\mu})$ *contractions in* $L_p(\hat{\nabla}, \hat{\mu})$, $1 < p < \infty$ *such that* $T_i\mathbb{1} \leq \mathbb{1}$ *for all* $i : 1 \leq i \leq d$, *and* $\alpha(\mathbf{k})$ *be a bounded Besicovich weights. Then for every* $\widehat{f} \in L_p(\hat{\nabla}, \hat{\mu})$, *the averages*

$$A_{\mathbf{n}}(\widehat{f}) = \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k=1}}^{\mathbf{n}} \alpha(\mathbf{k})\mathbf{T}^{\mathbf{k}}(\widehat{\mathbf{f}})$$

(o)*- converge to some* $\widetilde{f}$ *in* $L_p(\hat{\nabla}, \hat{\mu})$.

Proof. Using Theorem 2.2 we immediately obtain that

$$\mathbf{T}^{\mathbf{k}}(\widehat{f})(\omega) = (T_1^{k_1}\cdots T_d^{k_d}\widehat{f})(\omega) = T_1^{k_1}(\omega)\cdots T_d^{k_d}(\omega)f(\omega) = \mathbf{T}^{\mathbf{k}}(\omega)f(\omega)$$

for any $\widehat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ and for a.e. $\omega \in \Omega$. Hence,

$$A_{\mathbf{n}}(\widehat{f})(\omega) = \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} \alpha(\mathbf{k})\mathbf{T}^{\mathbf{k}}(\omega)(f(\omega)) = A_{\mathbf{n}}(f(\omega))$$

for a.e. $\omega \in \Omega$ and for any $\widehat{f} \in L_p(\hat{\nabla}, \hat{\mu})$.

Since every T_i is linear positive $L_1(\hat{\nabla}, \hat{\mu})$ contractions in $L_p(\hat{\nabla}, \hat{\mu})$ and $T_i\mathbb{1} \leq \mathbb{1}$ then by Theorem 2.2 every $T_i(\omega)$ is positive $L_1(\nabla_\omega, \mu_\omega)$ contraction in $L_p(\nabla_\omega, \mu_\omega)$ and $T_i\mathbb{1}(\omega) \leq \mathbb{1}(\omega)$ for a.e. $\omega \in \Omega$. This means that $T_i(\omega)$ is $L_1(\nabla_\omega, \mu_\omega) - L^\infty(\nabla_\omega, \mu_\omega)$- contraction. Then by Theorem 1.2 [1] the averages $A_{\mathbf{n}}(f(\omega))$ (o)-converge to some $g(\omega) \in L_p(\nabla_\omega, \mu_\omega)$. According Corollary 3.3 we find $A_{\mathbf{n}}(\widehat{f}) = \widehat{A_{\mathbf{n}}(f(\omega))} \stackrel{(o)}{\to} \widetilde{f} = \widehat{g(\omega)}$ in $L_0(\hat{\nabla}, \hat{\mu})$.

From

$$
\begin{aligned}
|A_{\mathbf{n}}(f(\omega))| &= |\frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} \alpha(\mathbf{k})\mathbf{T}^{\mathbf{k}}(\omega)(f(\omega))| \\
&\leq \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} |\alpha(\mathbf{k})|\mathbf{T}^{\mathbf{k}}(\omega)(|f(\omega)|) \\
&\leq \frac{b}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} \mathbf{T}^{\mathbf{k}}(\omega)(|f(\omega)|)
\end{aligned}
$$

and

$$
\sup_{\mathbf{n}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} \mathbf{T}^{\mathbf{k}}(\omega)(|f(\omega)|) \in L_p(\nabla_\omega, \mu_\omega),
$$

we find $\sup\limits_{\mathbf{n}} |A_{\mathbf{n}}(f(\omega))| \in L_p(\nabla_\omega, \mu_\omega)$, where $b = \sup\limits_{\mathbf{k}} |\alpha(\mathbf{k})|$ for a.e. $\omega \in \Omega$.

According Theorem 3.1 one gets $\sup\limits_{\mathbf{n}} |A_{\mathbf{n}}(\widehat{f})| \in L_0(\hat{\nabla}, \hat{\mu})$.

From Theorem 1.2 [15](p.196) it follows that

$$
\left\| \sup_{\mathbf{n}} |A_{\mathbf{n}}(f(\omega))| \right\|_p \leq b \left\| \sup_{\mathbf{n}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{k}=1}^{\mathbf{n}} \mathbf{T}^{\mathbf{k}}(\omega)(|f(\omega)|) \right\|_{L_p(\nabla_\omega, \mu_\omega)} \leq b \cdot q^d \|f(\omega)\|_{L_p(\nabla_\omega, \mu_\omega)}
$$

for a.e. $\omega \in \Omega$.

This means that

$$
\| \sup_{\mathbf{n}} |A_{\mathbf{n}}(\widehat{f})| \|_{L_p(\hat{\nabla}, \hat{\mu})} \leq b \cdot q^d \|\widehat{f}\|_{L_p(\hat{\nabla}, \hat{\mu})}
$$

and $A_{\mathbf{n}}(\widehat{f})$ is bounded in $L_p(\hat{\nabla}, \hat{\mu})$. Hence $A_{\mathbf{n}}(\widehat{f}) \stackrel{(o)}{\to} \widetilde{f}$ in $L_p(\hat{\nabla}, \hat{\mu})$. □

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